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Generalized M-Matrices and Applications

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Abstract. Recently, two distinct directions have been taken in an attempt to generalize the definition of an M-matrix. Even for nonsingular matrices, these two generalizations are not equivalent. The role of these and other classes of recently defined matrices is indicated showing their usefulness in various applications.

1. Introduction. All matrices considered are real. A square matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leqslant 0$ whenever $i \neq j$ and $A^{-1} \geqslant 0$. A is called *monotone* if $x \geqslant 0$ whenever $Ax \geqslant 0$. The usefulness of these matrices has been indicated in [2], [18] and [27].

Recently, two distinct directions have been taken in an attempt to generalize the definition of an M-matrix. Schneider [24] was responsible for the first direction and in addition to requiring the square matrix $A = (a_{ij})$ satisfy $a_{ij} \leqslant 0$ whenever $i \neq j$, he used the spectral properties of a nonsingular M-matrix to generalize to the singular M-matrix. The second direction is attributed to Plemmons [18] where he used the concept of monotonicity and the theory of generalized inverses to extend the definition of an M-matrix to include rectangular matrices. Even for nonsingular matrices these two generalizations are not equivalent.

The purpose of this paper is to compare these new definitions of an M-matrix together with the concept of monotonicity and to indicate their role in various applications.

2. History and Preliminaries. $A = (a_{ij})$ is called a *Stieltjes matrix* if $a_{ij} < 0$ whenever $i \neq j$ and A is a symmetric positive definite matrix. In 1887, Stieltjes [25] showed that such a matrix satisfied $A^{-1} > 0$. In 1912, Frobenius [9] proved the following stronger result: If $A = \alpha I - B$ where $B > 0$ and α exceeds the spectral radius of B, then $A^{-1} > 0$. In 1937, Ostrowski [16] generalized even further and his results together with those of several others are provided in the following definition.

Definition 2.1. Suppose $A = (a_{ij})$ satisfies $a_{ij} \leqslant 0$ whenever $i \neq j$ and $a_{ii} > 0$ for each i. The square matrix A is called an M-matrix if it satisfies any one of the following equivalent conditions [23]:

(a) $A = \alpha I - B$ for some nonnegative matrix B and some $\alpha > \rho$, where ρ is the spectral radius of B.

(b) The real part of each eigenvalue of A is positive.

(c) All principal minors of A are positive.

(d) A^{-1} exists and $A^{-1} \geqslant 0$.

(e) There exists a vector $x > 0$ such that $Ax > 0$.

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For additional information on M-matrices see [23]. We now consider several definitions which, due to properties (a), (d) and (e), are in some sense generalizations of an M-matrix. To place our later remarks in proper perspective, we have assumed that the classical definition of an M-matrix as given in Definition 2.1 is the only permissible one and that any generalization when restricted to the nonsingular case should agree with Definition 2.1.

Definition 2.2. Suppose $A = (a_{ij})$ satisfies $a_{ij} \leqslant 0$ whenever $i \neq j$ and $a_{ii} \geqslant 0$ for each i. The square matrix A is called an M-matrix if it satisfies any one of the following equivalent conditions [23]:

(a) $A = \alpha I - B$ for some nonnegative matrix B and some $\alpha \geqslant \rho$, where ρ is the spectral radius of B.

(b) The real part of each nonzero eigenvalue of A is positive.

(c) All principal minors of A are nonnegative.

Definition 2.3. A square matrix A is called monotone if it satisfies any one of the following equivalent conditions [7]:

(a) $Ax \geqslant 0$ implies $x \geqslant 0$.

(b) A^{-1} exists and $A^{-1} \geqslant 0$.

Actually, Mangasarian [10] has generalized monotonicity to include all matrices satsifying condition (a). However, we take Collatz's [7] definition as given above.

Definition 2.4. A rectangular matrix $A = (a_{ij})$ is called row monotone if it satisfies any one of the following equivalent conditions, [1] and [18]:

(a) $x \in R(A^T)$ and $Ax \geqslant 0$ implies $x \geqslant 0$. ($R(A)$, A^T denotes the range and transpose of A, respectively.)

(b) $AX \geqslant 0$ implies $AA^+X \geqslant 0$. (A^+ denotes the Moore-Penrose inverse of A [17].)

(c) The system $Y \geqslant 0$, $YA = A^+A$ is consistent.

(d) $A^+ = B + C$ for some $B \geqslant 0$ and C such that $CA = 0$.

Definition 2.5. A matrix A is called semimonotone if $A^+ \geqslant 0$.

We might remark that the problem of finding conditions for which a matrix A has a nonnegative generalized inverse A^g (where A^g may be a group inverse or one satisfying any one of a number of combinations of the four Penrose equations) has been researched in detail. A general report and appropriate references are contained in [3].

Definition 2.6. Suppose A has order (m, n) and can be expressed in the form $A = \alpha B - M$ where $M = BG \geqslant 0$, B has rank n, and $B^+ \geqslant 0$. Then A is called a rectangular M-matrix if A satisfies any one of the following equivalent conditions [18]:

(a) $\alpha \geqslant \rho$ where ρ is the spectral radius of G $(G = B^+M \geqslant 0)$.

(b) $A^+ \geqslant 0$.

Note that A has full column rank.

Definition 2.2 is due to Schneider (see [23] and [24]). Definition 2.4 is due to Berman and Plemmons [1], and Definition 2.6 is due to Plemmons [18]. It is primarily Definitions 2.2 and 2.6 we are interested in comparing. However, the relationships between all six of the definitions given above will be indicated giving a better perspective of the entire situation. To accomplish this, let M, M^-, M_0, M_r, M^+, M_+ denote the classes of matrices defined in Definitions 2.1–2.6, respectively.

The following notation is adopted:

A^T	The transpose
$R(A)$	The range
$N(A)$	The null space
A^{-1}	The inverse
A^+	The Moore-Penrose inverse
$\rho(A)$	The spectral radius
$A \geqslant 0$	The entries of A are nonnegative
0	The zero matrix
I	The identity matrix
$\subseteq$	Subset
$\subset$	Proper subset
(m, n)	Indicates the size (order) of a matrix

3. Relationships between $M, M^-, M_0, M_r, M^+, M_+$.

LEMMA 3.1. $M \subset M_0 \subset M_r$.

Proof. $M \subseteq M_0 \subseteq M_r$ follows from Definitions 2.1, 2.3 and 2.4. To show that $M \neq M_0$, consider

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, \quad \text{where } A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

$A \in M_0$ but $A \notin M$. To show that $M_0 \neq M_r$, consider $A = [1, 1]$. $A \in M_r$ but $A \notin M_0$.

LEMMA 3.2. $M_+ \subset M^+ \subset M_r$.

Proof. $M_+ \subseteq M^+ \subseteq M_r$ follows from Definitions 2.6, 2.5 and 2.4. The first matrix in the proof of Lemma 3.1 belongs to M^+ and not to M_+ so that $M^+ \neq M_+$. To show $M^+ \neq M_r$, consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \text{where } A^+ = \begin{bmatrix} 1/4 & -1/4 \\ 1/4 & -1/4 \end{bmatrix}.$$

$A \in M_r$ but $A \notin M^+$.

LEMMA 3.3. *$M \subset M_+$ and $M_0 \subset M^+$.*

Proof. The lemma follows from Definitions 2.1, 2.6, 2.3 and 2.5 and the fact that M_+ and M^+ contain nonsquare elements.

LEMMA 3.4. $M \subset M^-$. *Furthermore, when $A \in M^-$ and A is nonsingular, then $A \in M$* [23].

In view of Lemma 3.4, Definition 2.2 would be considered an acceptable generalization of an M-matrix. On the other hand, this same property does not hold for Definition 2.6. That is, even though $M \subset M_+$ (Lemma 3.3), M_+ contains nonsingular matrices which do not belong to M. For example, consider the first matrix A in the proof of Lemma 3.1. Let $G = 0, B = A, \alpha = 1$. Then $A = \alpha B - 0$ where $M = BG \geqslant 0, B^{-1} \geqslant 0$ and $\alpha > \rho(G)$. According to Definition 2.6, $A \in M_+$. However, $A \notin M$. Even though Definition 2.6 is not the kind of generalization one normally works toward, this generalization specifies a class of matrices which are of considerable importance in iteration schemes, as will be indicated in Section 4.

We might also note from the definitions that nonsingular row monotone matrices are also monotone.

LEMMA 3.5. *There is no set inclusion between M^- and M_0, M^- and M_r, M^- and M_+, M^- and M^+.*

Proof. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2I - B = 2I - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

where $2 = \rho(B)$ and

$$A^+ = \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix}.$$

Then $A \in M^-$, $A \notin M_0$, $A \notin M^+$ and $A \notin M_+$.

Also the matrix

$$A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

belongs to M^- and not to M_r. We have established that $M^- \not\subset M_0$, $M^- \not\subset M_r$, $M^- \not\subset M_+$ and $M^- \not\subset M^+$.

Now the first matrix in Lemma 3.1 belongs to M_0, M^+ and M_+ but not to M^-. The matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

belongs to M_r and not to M^-. This shows that $M_0 \not\subset M^-$, $M^+ \not\subset M^-$, $M_+ \not\subset M^-$ and $M_r \not\subset M^-$ which establishes the lemma.

These five lemmas verify the following theorem indicating the relationship among those matrices defined in Definitions 2.1–2.6.

THEOREM 3.6. *The following diagram indicates the relationship among the classes M, M^-, M_0, M_r, M^+ and M_+:*

$$\begin{array}{lll} M^- & & \\ \cup & & \\ M & \subset M_0 & \subset M_r \\ \cap & \ \ \cap & \ \ \| \\ M_+ & \subset M^+ & \subset M_r \end{array}$$

4. Convergent Splittings. The classes of matrices we have considered arise in investigations concerning the convergence of iteration processes in matrix computation.

For example, consider the matrix equation

$$Ax = b \tag{4.1}$$

where A has order (m, n). When $m = n$ and A is nonsingular, many iteration techniques [27] for solving (4.1) can be obtained by splitting A into the difference of two matrices M and N and using the iteration

$$x_{i+1} = M^{-1}Nx_i + M^{-1}b. \tag{4.2}$$

This iteration converges for each x_0 if and only if $\rho(M^{-1}N) < 1$. Conditions on A, M, N to insure that $\rho(M^{-1}N) < 1$ were given by Varga [27] ($A^{-1}, M^{-1}, N \geqslant 0$) and Mangasarian [11] ($A^{-1}N, M^{-1}N \geqslant 0$).

If $A \in M$, then by setting certain off diagonal elements of A equal to zero and defining the resulting matrix as M so that $M^{-1} \geqslant 0$ [27 Theorem 3.12], then $N = M - A$ and the conditions of Varga are satisfied [27, Theorem 3.14]. Also, this method yields a nontrivial splitting ($N \neq 0$) whenever A is not diagonal.

If A is the first matrix in Lemma 3.1, then

$$A = M - N = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $A \notin M$ ($A \in M_0$) and $M^{-1} \geqslant 0$. Therefore $A^{-1}N \geqslant 0, M^{-1}N \geqslant 0$ and the conditions of Mangasarian are satisfied. Is there always a nontrivial splitting for elements of M_0 which satisfy the conditions of Mangasarian?

Generalizing these techniques, Berman and Plemmons ([1], [2] and [19]) have shown how to solve (4.1) using iterative techniques similar to (4.2) when $A \in M_+$, for example. To illustrate the general flavor of these techniques, we present the following theorem and proof.

THEOREM 4.3 (PLEMMONS). *Suppose that $A = M - N$ is a splitting for A and $\rho(M^+N) < 1$, $R(M^+) \subseteq R(A^T)$, $R(N) \subseteq R(M)$ and $MM^+ = AA^+$, then the iteration*

$$x_{i+1} = M^+Nx_i + M^+b \tag{4.4}$$

converges to the best least-squares approximate solution A^+b of (4.1), for any x_0.

Proof. Since $\rho(M^+N) < 1$, the iteration (4.4) converges to some vector y. Then

$$y = M^+Ny + M^+b \in R(M^+) \subseteq R(A^T)$$

and $(I - M^+N)y = M^+b$. Since $R(N) \subseteq R(M)$, (MM^+ is a projection on $R(M)$), $MM^+N = N$. Hence, $Ay = (M - N)y = (M - MM^+N)y = M(I - M^+N)y = MM^+b$. Also, $MM^+ = AA^+$ so that $Ay = AA^+b$. Since $y \in R(A^T)$, (A^+A is a projection on $R(A^T)$), $y = A^+Ay = A^+AA^+B = A^+b$, which completes the proof.

Plemmons has developed this technique in several papers (see [2], [18], [19], [20] and [21]) with slight variations and modifications depending on initial conditions of the given system $Ax = b$, such as consistency (underdetermined) or inconsistency (overdetermined), etc.

The interesting problem at this point is to recognize matrices A which possess nontrivial splittings $A = M - N$ satisfying the conditions in the hypothesis of Theorem 4.3.

One approach to this problem is to consider various well-defined classes of matrices and determine if they meet the conditions of Theorem 4.3. For example, the classes M, M_0 and M_+ satisfy these conditions as demonstrated below [18].

THEOREM 4.5 (PLEMMONS). *Suppose A has full column rank. Then $A^+ \geqslant 0$ if and only if there exists a matrix N such that*

(a) $M = A + N$ *satisfies* $R(N) \subseteq R(M)$ *and* $M^+ \geqslant 0$,

(b) $M^+N \geqslant 0$,

(c) $\rho(M^+N) < 1$.

A second approach is to generalize existing definitions so as to enlarge the class of matrices to which Theorem 4.3 may be applied. This appears to have been Plemmons' purpose in defining the class M_+, although it does not generalize the definition of a nonsingular M-matrix. Another class of matrices is defined as follows and is implicitly due to Plemmons [19].

Definition 4.6. A is a P-matrix if there exists a splitting $A = M - N$ such that

(a) $R(M) = R(A)$,

(b) $N(M) = N(A)$,

(c) $A^+N, M^+N \geqslant 0$.

Plemmons showed in [19] that necessarily $\rho(M^+N) < 1$. Theorem 4.3 applies to such matrices. Furthermore, if M_p denotes this new class of matrices, M_p contains matrices not included in M_0 or M_+. For example, consider

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = M - N,$$

where

$$A^+ = \begin{bmatrix} ½ & ½ \\ 0 & 0 \end{bmatrix} \geqslant 0, \qquad A \in M_p,$$

but $A \notin M_0$ and $A \notin M_+$.

In summary, the best least-squares solution to $Ax = b$ can be obtained through iteration techniques of the form (4.4), whenever A is a member of M_0, M_+ or M_p.

5. Nonnegative Solutions to Matrix Equations. Monotone matrices A quite often appear in matrix equations of the form $Ax = b$ when deriving finite difference approximations to solutions of certain elliptic differential equations (see the bibliography in [27]). Until [1], only matrices $A \in M_0$ were considered. Even though it is conceivable that singular matrices A in M^- might appear in $Ax = b$, the iteration technique in (4.4) does not apply in general. However, such matrices do appear in other situations and play a valuable role.

Rather than seek an iterative scheme to solve the equation $Ax = b$, one might be more interested in the problem of finding a nonnegative solution x when $b \geqslant 0$ (or even for arbitrary b). Such problems arise in the study of periodic solutions of linear elliptic partial differential equations which are not symmetric [26]. The study of nonnegative solutions is also important in statistics and linear programming problems (see [8], [14], [15]).

From our observations in the preceding sections, it is clear that the matrix equation $Ax = b$ and the matrix inequality $Ax \geqslant b$ (when $b \geqslant 0$) has nonnegative best least-squares approximate solutions whenever $A \in M^+$. Furthermore, if $b = 0$ and $A \in M_r$, then all solutions of $Ax = 0$ or $Ax \geqslant 0$ are nonnegative. Question: If $b \neq 0$, $b \geqslant 0$ and $A \in M_r$, when will $Ax = b$ or $Ax \geqslant b$ have nonnegative solutions?

As we have indicated, elements of M^- need not belong to M^+. However, Carlsor [5] provided necessary and sufficient conditions for the system $Ax = b$ $(b \geqslant 0)$, wher $A \in M^-$, to have a unique nonnegative solution. Rather than state his result which requires additional definitions and notation, we refer the reader to [5]. However, we note that these conditions of Carlson depend primarily on the location of zeros and nonzero elements (not their values) in A and b.

We close this section with an example indicating the usefulness of the class M^- and the results of Carlson where the material in Section 4 does not apply.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } A^+ = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $b = [1, 1, 1]^T$, it is not immediately clear that the equation $Ax = b$ has a nonnegative solution since A^+ is nonnegative. However, $A^+b \geqslant 0$ is the desired solution.

We remind the reader again of our remark immediately following Definition 2.5. λ-monotone matrices are defined in [3], and such matrices could prove valuable when seeking nonnegative solutions to the system $Ax = b$ depending on initial conditions.

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